Spinors, $SU(2)$, the Neutrino Masses, and All That Stuff

I. NOTES FROM ANDRÉ DE GOUVÊA'S LECTURE ON NEUTRINO MASS MODELS (TASI 2020)

(Adrian Thompson)

A. Spinors and Fermion Masses: Majorana or Dirac?

We first introduce Dirac fermions; ψ is a four-component, anti-commuting field whose degrees of freedom are identified as follows;

$$\left.\begin{array}{r}\text{Particle - 2 spin components}\\ \text{Anti-particle - 2 spin components}\end{array}\right\}\text{4 degrees of freedom} \tag{1}$$

The Dirac spinors are formally representations of a clifford algebra sometimes denoted $CL(1,3)$, which has generators γ of the algebra that act on the spinor reps.[1]

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{2}$$

We have to choose a matrix representation for the γ's that maintains their Dirac algebra. **Here we choose "Weyl" or "Chiral" representation.** In block form,

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3}$$

The σ^i's are the Pauli matrices, σ^1, σ^2, σ^3. For future reference in the discussion ahead, we state them here.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

In 4-dimensions, we also define $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. This matrix has the properties

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{5}$$

$$(\gamma^5)^2 = I \tag{6}$$

$$\tag{7}$$

and in our Weyl representation, we have

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{8}$$

The γ^5 matrix is interesting in particular, because it's eigenstates are those that we identify as the L and R chiral fields inside ψ;

$$\gamma^5\psi_L = -\psi_L \tag{9}$$

$$\gamma^5\psi_R = \psi_R \tag{10}$$

where we have $\psi = \begin{pmatrix} \chi \\ \xi \end{pmatrix}$, so we notate $\psi_L = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$ and $\psi_L = \begin{pmatrix} 0 \\ \xi \end{pmatrix}$ where each field χ, ξ are 2-component Weyl spinors; $\chi = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}$, $\xi = \begin{pmatrix} \xi_\uparrow \\ \xi_\downarrow \end{pmatrix}$. **We care about these Weyl spinors because they are the irreducible spin-$\frac{1}{2}$**

[1] There should also be a connection here between $CL(1,3)$ amd $SL(2,\mathbb{C})$, but I need to solidify my understanding of that more - perhaps that should be saved for another set of notes.

reps of the Lorentz group[2] In most reps, the top 2 components of the 4-component spinor are identified with particles, and the bottom 2 components with anti-particles. **But in our Chiral representation, χ is identified with left-chiral fermions and ξ with right-chiral fermions.**

We can write down the Lorentz scalars that we can build out of ξ and χ in the kinetic and mass terms of our Lagrangian;

$$\mathcal{L} \supset i\xi^\dagger \partial_\mu \sigma^\mu \xi + i\chi^\dagger \partial_\mu \bar{\sigma}^\mu \chi - m(\xi^\dagger \chi + \chi^\dagger \xi) \tag{11}$$

where we have constructed a 4-vector out of the γ's;

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{12}$$

for $\sigma^\mu = (I, \sigma^i)$, $\bar{\sigma}^\mu = (I, -\sigma^i)$. Notice that in Eq. 11, if we go to the massless limit, the left and right components ξ and χ would decouple and we could not tell any physical difference between these two types of spinors. The mass term distinguishes them. Also, if we were to introduce a gauge symmetry with new fields for the ξ's and χ's to interact with, the kinetic terms would take $\partial_\mu \to D_\mu$ and ξ and χ would still be separate as long as we have the mass term.

In short, we can recapitulate the statement 14 as

$$\chi = \begin{cases} \text{left-helicity particle} \\ \text{right-helicity anti-particle} \end{cases} \tag{13}$$

$$\xi \equiv \begin{cases} \text{right-helicity particle} \\ \text{left-helicity anti-particle} \end{cases} \tag{14}$$

The SM is made out of chiral fermions; it is a "chiral gauge theory". Everything in our Lagrangian can be written in terms of χ and ξ.

Now we can introduce the **charge conjugate**[3] ψ^c;

$$\psi^c \equiv \underbrace{-\eta_c}_{\pm 1} \gamma^0 C \psi^* \tag{15}$$

Here the factor η_c is a parity factor, and in our representation, the charge conjugation operator $C \equiv i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$. If we write out what this does to ψ, we get

$$\psi^c = -\eta_c \begin{pmatrix} -i\sigma_2 \xi^* \\ i\sigma_2 \chi^* \end{pmatrix} \equiv \begin{pmatrix} \chi^c \\ \xi^c \end{pmatrix} \tag{16}$$

From the result of the transformation above, we have identified $\chi^c = \eta_c i\sigma_2 \xi^*$. Therefore $-i\sigma_2\xi^*$ is the LH component of the charge-conjugate field; $\xi = i\sigma_2(\chi^c)^*$. Symbolically,

$$\xi \iff \chi^c \tag{17}$$

This is why we identify the upper components of ξ and χ with particles, and the lower components with anti-particles; the action of $i\sigma_2$ on ξ and χ exchanges the places of the upper and lower components. From Eq. 16 we are allowed to write everything in terms of one symbol, χ and its charge conjugate χ^c;

$$\psi = \begin{pmatrix} \chi \\ i\sigma_2(\chi^c)^* \end{pmatrix} \tag{18}$$

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[2] Note: see http://www.weylmann.com/weyllorentz.pdf

[3] This is a good resource to learn about charge conjugation: https://sites.ualberta.ca/~gingrich/courses/phys512/node64.html

The Lagrangian is reexpressed as

$$\mathcal{L} = i\chi^\dagger \partial_\mu \bar{\sigma}^\mu \chi + i(\chi^c)^\dagger \partial_\mu \bar{\sigma}^\mu \chi^c - m[(\chi^c)^T (i\sigma_2)\chi + \underbrace{\chi^\dagger (i\sigma_2)(\chi^c)^*}_{\text{h.c. of the first term}}] \tag{19}$$

where the combinations

$$(\chi^c)^T (i\sigma_2)\chi = \chi^c_\uparrow \chi_\downarrow - \chi^c_\downarrow \chi_\uparrow \tag{20}$$

are spin-0 combinations; these are what the mass terms should look like. The mass terms are the sum of all the spin-0 combinations.

So, we can make Lorentz scalars out of $\chi \cdot \chi^c \equiv \chi^c_\uparrow \chi_\downarrow - \chi^c_\downarrow \chi_\uparrow$. But what about terms like $\chi \cdot \chi$? **This brings us to the 2 different types of mass terms;**

Dirac Mass: $m_D \chi \cdot \chi^c$

Majorana Mass: $m_M \chi \cdot \chi$

Note that $\chi \cdot \chi = \chi_\uparrow \chi_\downarrow - \chi_\downarrow \chi_\uparrow \neq 0$ since they anticommute; $\{\chi_\uparrow, \chi_\downarrow\} = 0$. If we only had χ[4], in the Majorana case, our Lagrangian would become

$$\mathcal{L} = i\chi^\dagger \partial_\mu \bar{\sigma}^\mu \chi - \frac{1}{2} m \chi \cdot \chi + \text{h.c.} \tag{21}$$

We will discuss specific instances of Majorana mass models in the next two sections.

As a last aside, it's good to mention here that we can still make a 4-component object out of χ's alone; this looks like

$$\psi = \begin{pmatrix} \chi \\ (i\sigma_2)\chi^* \end{pmatrix} \tag{22}$$

which *looks* Dirac, but really only has two unique components ($\chi_\uparrow$ and $\chi_\downarrow$). We could call this a "Majorana Spinor."

B. Fermion Masses in the SM: Neutrino Mass Models

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	
Q	3	2 $\begin{pmatrix} u \\ d \end{pmatrix}$	$+1/6$	
u^c	$\bar{3}$	1	$-2/3$	$\times 3$ generations
d^c	$\bar{3}$	1	$+1/3$	
L	1	2 $\begin{pmatrix} \nu \\ e \end{pmatrix}$	$-1/2$	
e^c	1	1	$+1$	

TABLE I. SM fermions and their charges

We tabulate the representations and charges of the SM fermions in Table I.

- When the fermions get charged under the gauge symmetries, this takes $\partial_\mu \to D_\mu$.
- The hypercharges are chosen such that the EM charges q are correct after electroweak symmetry breaking (EWSB) breaks $SU(2)_L \times U(1)_Y$.

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[4] In SUSY, the superpotential has to be "holomorphic". It is easier to demand this when we only work with L or R chiral fields $\to$ *AT: I suppose this suggests Majorana fermions are more fitting in SUSY scenarios.*

- There are no bilinears that we can write from this table that are gauge invariant → **we need the Higgs field to do this**.

We introduce a higgs field with notation

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix} \quad \tilde{H} = \begin{pmatrix} (h^0)^* \\ -h^- \end{pmatrix} \tag{23}$$

where H gets hypercharge $Y = 1/2$ and $\tilde{H}$ gets $Y = -1/2$. The Yukawa couplings for one family take the form

$$\mathcal{L}_{Yuk} \supset \lambda_u Q u^c H + \lambda_d Q d^c \tilde{H} + \lambda_e L e^c \tilde{H} + \text{h.c.} \tag{24}$$

Here we have to explain some notation; when we write the product QH, this is shorthand for $uh^0 - dh^+$ since we are really contracting with $i\sigma_2$. Writing it out explicitly, we would have

$$QH \equiv Q^T(i\sigma_2)H = \begin{pmatrix} u \\ d \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} h^+ \\ h^0 \end{pmatrix} \tag{25}$$

Also, note that Qu^c is manifestly Lorentz invariant since it involves the combinations $\chi\chi^c$ as per what we learned in the last section.

Assigning a VEV to H the following way, we can generate the fermion masses;

$$\langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad \langle \tilde{H} \rangle = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \tag{26}$$

However, this predicts massless neutrinos, which we now know is wrong. From oscillation experiments, we know that at least 2 of the neutrino masses lie in the range $[0.01, 0.1]$ eV, but this is separated from the rest of the fermion masses by 6 orders of magnitude (the weak scale). Why are they so small? It's a puzzle;

- We need new degrees of freedom in the SM to answer this
- There are **lots** of choices
- How do we find the right choice?

The first question is whether the neutrinos have Majorana masses or Dirac masses. In the SM, the neutrinos ν live in the doublets L which couple to the higgs field, so the way they acquire masses will necessarily involve EWSB.

Let us first consider the case of Dirac masses. We could add a ν^c to partner up with the ν's to give us a Dirac mass, with terms like

$$\mathcal{L} \supset \lambda_\nu L \nu^c H \tag{27}$$

where ν^c is a $(1, 1, 0)$ representation of $SU(3)_C \times SU(2)_L \times U(1)_Y$. This model would be a simple way to give the neutrinos a mass - why don't we use it? Because the combination $\nu^c\nu^c$ is also Lorentz and Gauge invariant, so nothing would prevent having terms like

$$\mathcal{L} \supset \frac{m}{2}\nu^c\nu^c \tag{28}$$

which is **not** a Dirac mass term → these can't really be Dirac fermions! We need to have a consistent theory.

The answer must then come from demanding a symmetry structure that gives us a consistent picture of the neutrino sector and its masses. The recipe for how to do this in the SM is pictured below:

After constructing the Lagrangian in the SM, there are 2 anomalous global symmetries:

$$\left.\begin{matrix} U(1)_B \\ U(1)_L \end{matrix}\right\} U(1)_{B-L} \tag{29}$$

We get $U(1)_{B-L}$ for free. If we write down a generic BSM, we can break $B - L$. We have good reason to do this, namely that there is a result from quantum gravity that does not "allow" global symmetries.[5] One way of breaking $B - L$, for example, is with the interaction we wrote previously;

$$-\mathcal{L} \supset \lambda_\nu L \nu^c H + \frac{m}{2}\nu^c\nu^c + \text{h.c.} \tag{30}$$

[5] It would be good to add a reference here for this - weak gravity conjecture.

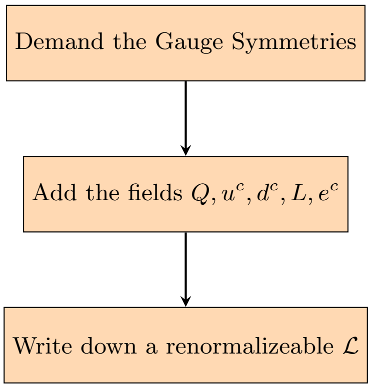

FIG. 1. The SM recipe

C. The Weinberg Operator: UV completions

The generic BSM effective operator that parameterizes the generation of the neutrino masses is the dim-5 **Weinberg operator**. Symbolically, contracting over the appropriate indices, it is

$$\frac{1}{2\Lambda}(LH)(LH) \tag{31}$$

We can learn about the Majorana or Dirac nature of the neutrinos from this operator, but not much more. For that, we would need to know the UV structure that gives rise to this higher dimension operator. By analogy, this is like how we studied the weak interactions through the Fermi theory of effective lepton interactions, like

$$\mathcal{L}_{\text{Fermi}} \supset G_F \bar{\mu}\gamma_\alpha \nu_{\mu,L} \bar{\nu}_e \gamma^\alpha e_L \,, \tag{32}$$

for which we needed "new physics" to support, i.e. the W boson. But the Fermi operator by itself didn't tell us how big M_W was by itself. That is where we had to test the UV completion of the Fermi model (electroweak theory) at the LHC and in muon decays etc.

We now turn to a few examples of UV completions of the Weinberg operator; Type-II and Type-I seesaw models.

1. Type II Seesaw

In the type-II seesaw model, we arrange the leptons into a triplet with hypercharge -1:

$$LL \to \begin{pmatrix} \nu\nu \\ (\nu e + e\nu)/\sqrt{2} \\ ee \end{pmatrix} \tag{33}$$

and we add a scalar triplet higgs field T with hypercharge $+1$:

$$T = \begin{pmatrix} t^{++} \\ t^{+} \\ t^{0} \end{pmatrix} . \tag{34}$$

Then we could suppose interactions of the form $\mathcal{L} \supset \lambda_T LLT$. From this setup, we can speculate that if T takes on a VEV in the lowest component;

$$\langle T \rangle = \begin{pmatrix} 0 \\ 0 \\ v_T \end{pmatrix} , \tag{35}$$

then the VEV of T will contribute to the generation of the lepton masses.

AT: I'm not totally sure how LLT is contracted in de Gouvêa's notation above. But if we represent T as a 2x2 matrix instead of a column vector,

$$T = \begin{pmatrix} t^+ & t^{++} \\ t^0 & -t^+ \end{pmatrix},$$

and L in its usual doublet form, then the contraction $L^T(i\sigma_2)TL$ seems to work.

With this construction, after EWSB $\langle T \rangle$ would give rise to the mass term $v_T \nu\nu$. This also spontaneously breaks $B - L$, which would give rise to a Goldstone boson - a massless scalar or pseudoscalar. This is bad, in a sense, because this Goldstone, φ let's say, would contribute to pion decay $\pi^0 \to \mu^+ \bar{\nu} \varphi$ which we can rule out.

Then, instead of the $\lambda_T LLT$ interaction, our solution will have to be a potential term like

$$V \supset k(\tilde{H}T\tilde{H}) \to \text{ explicit } \not{L} \text{ with no Goldstone} \tag{36}$$

Now v and v_T are the VEVs which contribute to the W, Z masses, and we would take $v_T << v$. This will allow a small neutrino mass; from the triplet model, we get mass terms from the yukawa matrices

$$\lambda_T^{ij} L_i L_j T \to m_\nu^{ij} = \lambda_T^{ij} v_T \tag{37}$$

In diagramatic form, our Weinberg operator is shown in Fig. 2 which gets integrated out to give us

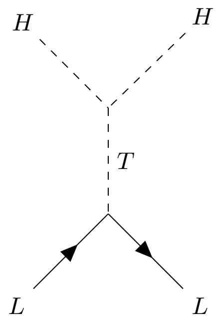

FIG. 2. Type II Seesaw

$$\sim \frac{(LH)(LH)}{\Lambda} \tag{38}$$

We call this a "seesaw" model because when you integrate out the diagram to get the operator above, it turns out that

$$\frac{1}{\Lambda} \propto \frac{1}{m_T^2}$$

.

2. Type I Seesaw

In type-II seesaw models, we would add ν^c back into the story - we might identify them with the notion of "sterile" neutrinos.

$$-\mathcal{L} = \lambda_\nu L \nu^c H + \frac{M}{2} \nu^c \nu^c \tag{39}$$

What is the effect of the mass M? In this scenario,

- M adds quantum corrections

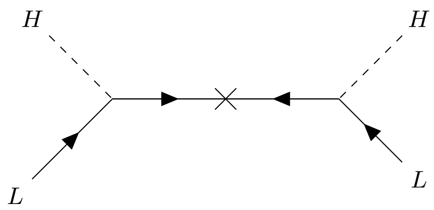

FIG. 3. Type I Seesaw

- M breaks lepton number symmetry
- but it has nothing to do with the weak scale or the higgs VEV

So what if M is much bigger than the weak scale? We can integrate out the weak scale and generate our Weinberg operator via the diagram in Fig. 3.

After EWSB, our Lagrangian will look like

$$\mathcal{L} \supset m_D \nu \nu^c + \frac{M}{2} \nu^c \nu^c \tag{40}$$

where m_D is the Dirac mass. This could be expressed in terms of a mass matrix

$$\mathcal{L} \supset \begin{pmatrix} \nu \\ \nu^c \end{pmatrix}^T \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix} \begin{pmatrix} \nu \\ \nu^c \end{pmatrix}. \tag{41}$$

Diagonalizing this mass matrix to see the neutrino mass eigenstates generates a heavy neutrino with mass $m_\nu^{\rm heavy} \sim M$ while $m_\nu^{\rm light} \sim m_D^2/M$. Since M is more or less unconstrained, this gives us the freedom to take heavy $M >> m_D$ to generate very light neutrino masses but with heavy sterile masses.

II. SU(2): CHANGE OF BASES FROM TRIPLET TO DOUBLET REPRESENTATION

If we have a triplet field Φ under $SU(2)$, we can expand Φ in the basis of Pauli matrices;

$$\Phi \to x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \tag{42}$$

for field degrees of freedom x_1, x_2, x_3. Writing it out in matrix form, we have

$$\Phi = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \tag{43}$$

We act with the analog of the charge operator in this new basis;

$$\Delta \equiv [\Phi, \tau_3] + \frac{Y}{2}\Phi \tag{44}$$

where $\tau_3 = \sigma_3/2$. Expanding out the new field Δ in matrix form we see that

$$\Delta = \begin{pmatrix} x_3 Y/2 & x_1 - ix_2 + (x_1 - ix_2)Y/2 \\ -x_1 - ix_2 + (x_1 + ix_2)Y/2 & -x_3 Y/2 \end{pmatrix} \tag{45}$$

If Φ has a hypercharge of $Y = 2$, then we have

$$\Delta = \begin{pmatrix} x_3 & 2(x_1 - ix_2) \\ 0 & -x_3 \end{pmatrix} \tag{46}$$

This motivates the definitions

$$\Phi \equiv \begin{pmatrix} \delta^+ & \delta^{++} \\ \delta^0 & -\delta^+ \end{pmatrix}, \quad \Phi^c \equiv \begin{pmatrix} \delta^- & \delta^{--} \\ \delta^0 & -\delta^- \end{pmatrix} \tag{47}$$

which allows us to find Lorentz-invariant combinations of the field Φ with other fields that respect $SU(2)$ gauge invariance, i.e. combinations that when expanded only leave charge-neutral terms. For instance, if we have a doublet

$$H \equiv \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}, \quad H^c \equiv \begin{pmatrix} h^- \\ h^0 \end{pmatrix} \tag{48}$$

then taking the product

$$\begin{aligned}(H^c)^T(\Phi^c)^T\Phi H = \delta^-\delta^+(h^0)^2 + \delta^{--}\delta^{++}(h^0)^2 - \delta^0\delta^+h^0h^- + \delta^-\delta^{++}h^0h^- \\ - \delta^0\delta^-h^0h^+ + \delta^{--}\delta^+h^0h^+ + (\delta^0)^2h^-h^+ + \delta^-\delta^+h^-h^+\end{aligned} \tag{49}$$

we can see easily that each term in the expansion is charge neutral simply by virtue of the $+, -$ notation indicating the eigenvalues of the charge operator.

Now the challenge is to ask "what are all the Lorentz scalars we can make out of Φ and other fields that are not only charge neutral, but also $SU(2)_L \times U(1)_Y$ singlets?"

To construct a $U(1)_Y$ singlet, We can combine the triplet field with two $Y = 1$ doublets. One of the possible $Y = -1$ doublet is the Lepton doublet L. $\bar{L}$ on the other hand is a $Y = 1$ doublet because of the complex conjugation. However, $\bar{L^c}$ involves a double conjugation of the lepton doublet field. Therefore, it is a $Y = -1$ doublet. Therefore the mass terms of this type is:

$$\mathcal{L}_M \sim \bar{L^c}\Phi L \tag{50}$$

Now the question arises as to whether it is $SU(2)_L$ invariant.

III. MAJORANA REPRESENTATION AND MAJORANA FERMIONS

(Aparajitha Karthikeyan)

The spinor fermion fields are goverened by the Dirac equation:

$$(i\gamma^{\mu}\partial_{\mu} - m)\Psi = 0 \tag{51}$$

γ^{μ} are $4x4$ matrices that must satisfy the Clifford Algebra (2) and

$$\gamma_0\gamma_{\mu}\gamma_0 = \gamma_{\mu}^{\dagger} \tag{52}$$

We came across Dirac or Weyl representations for the Gamma mateices. One another possible representation is the Majorana representation where all the Gamma matrices are purely imaginary.

$$\tilde{\gamma_0} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \tilde{\gamma_1} = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix} \tilde{\gamma_2} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \tilde{\gamma_3} = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \tag{53}$$

One noticeable feature of this representation is that the operator acting on the fermion field in the Dirac equation (51) is Real, or hermitian. Therefore it is possible to obtain a real solution for Ψ of the Dirac equation. Let us denote these real solutions with a tilde, $\tilde{\psi}$ such that

$$\tilde{\psi}^* = \tilde{\psi} \tag{54}$$

Such a solution to the Dirac equation is called a Majorana fermion. Note: It is not always possible to obtain a real solution for the Dirac equation. Since there are infinite number of choices for the Dirac matrices, all satisfying (2) and (52), there always exists a unitary transformation between two representations of Gamma matrices. in other words, if γ^{μ} (without the tilde) are the matrices in any random representation, they can be related to $\tilde{\gamma^{\mu}}$ by

$$\gamma^{\mu} = U\tilde{\gamma^{\mu}}U^{\dagger} \tag{55}$$

Accordingly, the solution of the Dirac equation in this general representation would be

$$\Psi = U\tilde{\Psi} \tag{56}$$

The reality condition (54) expressed in the new representation would be

$$U^{\dagger}\Psi = (U^{\dagger}\Psi)^* = U^T\Psi^* \tag{57}$$

$$\Psi = UU^T\Psi^* \tag{58}$$

To put it formally, let us define the Charge conjugation operator here as

$$\gamma_0 C = UU^T \tag{59}$$

And a charge conjugate state as

$$\Psi^c = \eta_c\gamma_0 C\Psi^* \tag{60}$$

Then if it is a Majorana fermion, the reality condition (54) in our chosen representation is

$$\Psi^c = \Psi \tag{61}$$

At this point in time, we are bound to have the following two questions. 1. For any fermion, we can find a real solution in the Majorana representation. So why can't all fermions be Majorana fermions? 2. Why is $\gamma_0 C$ specifically called the charge conjugation operator? To answer the first question, we need to note that real solutions are possible only for chargeless fermions. For a charged fermion, the normal derivative becomes a Covariant derivative. Then the Dirac equation becomes

$$(i\tilde{\gamma}^{\mu}D_{\mu} - m)\Psi = (i\tilde{\gamma}^{\mu}\partial_{\mu} + q\tilde{\gamma}^{\mu}A_{\mu} - m)\Psi = 0 \tag{62}$$

In other words, for a charged field, the coupling of Ψ and A_μ contributes to the Dirac equation. Even if we get into the purely imaginary Majorana representation of the Gamma matrices, this term is purely imaginary. Therefore the Dirac equation is not real anymore and it becomes impossible to obtain real solutions for a charged fermion field Ψ.

Coming to the second question, the complex conjugate of (62) looks like

$$(i\tilde{\gamma}^\mu \partial_\mu - q\tilde{\gamma}^\mu A_\mu - m)\Psi^* = 0 \tag{63}$$

This shows that Ψ^* is a fermion field with charge $-q$. In other words, the charge conjugation operation in the Majorana representation is complex conjugation

$$\tilde{\Psi}^c = \tilde{\Psi}^* \tag{64}$$

When we go into any other representation, $\gamma_0 C$ along with the complex conjugation essentially performes the same operation that the complex conjugation performed in the Majorana representation. Therefore, $\gamma_0 C$ is the charge conjugation operator and Ψ^c is the charge conjugate field.

One can verify that the Unitary matrix that can take us from the Majorana representation to the Weyl representation is

$$U = \frac{1}{2}\begin{pmatrix} 1+\sigma_2 & -i(1-\sigma_2) \\ i(1-\sigma_2) & 1+\sigma_2 \end{pmatrix} \tag{65}$$

Therefore,

$$\gamma_0 C = UU^T = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \tag{66}$$

Or,

$$\gamma_0 C = UU^T = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = i\gamma^2 \tag{67}$$